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On the Outer and Inner Invariants of Connes

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1. INTRODUCTION

Let $\mathfrak{A}$ be the approximately finite II_1 factor of Murray and von Neumann. The group of all *-automorphisms of $\mathfrak{A}$ is denoted Aut $\mathfrak{A}$. Let α, β be elements of Aut $\mathfrak{A}$. Then α is *conjugate* to β, written $\alpha \sim \beta$, if $\alpha = \sigma\beta\sigma^{-1}$ for some σ in Aut $\mathfrak{A}$. The automorphism α is *periodic* if $\alpha^n = 1$ for some positive integer n.

A complete set of invariants for the conjugacy classes of periodic elements of Aut $\mathfrak{A}$ was recently found by Connes [4]. These invariants are denoted p_0, γ, and ϵ where

(i) p_0 is a positive integer,

(ii) γ is a scalar of modulus 1 such that $\gamma^{p_0} = 1$,

(iii) ϵ is a probability measure in the qth roots of unity, for some positive integer q.

The pair p_0, γ is called the *outer invariant*; ϵ is the *inner invariant.*

Let E be real separable infinite-dimensional Hilbert space and let j be a linear map from E to the self-adjoint part of $\mathfrak{A}$ such that

(a) $j(x)^2 = \| x \|^2 \cdot 1$ for all vectors x in E,

(b) $\mathfrak{A}$ is generated by the image of j.

Such maps certainly exist [2]. Let

$$\mathrm{Aut}_E(\mathfrak{A}; j) = \{\phi \in \mathrm{Aut}\ \mathfrak{A} \colon \phi(j(E)) \subset j(E)\}.$$

This is the subgroup of *Bogoliubov* automorphisms of $\mathfrak{A}$, relative to the map j. This terminology originates with the quantum theory of the free fermion field: see, for example, Balslev, Manuceau, and Verbeure [1].

Let j_1 and j_2 be two maps satisfying conditions (a) and (b). Now $\mathfrak{A}$ is the *von Neumann–Clifford* algebra over $j_1(E)$ and $j_2(E)$, each of which is a copy of E. Consequently, there exists a unique automorphism σ of $\mathfrak{A}$ such that $j_2 = \sigma \circ j_1$. Then $\phi \in \mathrm{Aut}_E(\mathfrak{A}; j_1)$ if and only if $\sigma\phi\sigma^{-1} \in \mathrm{Aut}_E(\mathfrak{A}; j_2)$ so that $\mathrm{Aut}_E(\mathfrak{A}; j_1)$ and

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$\mathrm{Aut}_E(\mathfrak{A}; j_2)$ are conjugate subgroups. Therefore the subgroup of Bogoliubov automorphisms of $\mathfrak{A}$ is determined up to conjugacy. The von Neumann–Clifford algebra is studied in great detail by Schrader and Uhlenbrock [6, Proposition 3.16]. The *Clifford group* as defined by Bourbaki [3, Section 9] is essentially the classical analog of the group of Bogoliubov automorphisms.

From now on, we shall denote the subgroup of Bogoliubov automorphisms by $\mathrm{Aut}_E(\mathfrak{A})$. It is clear that

$$\mathrm{Aut}_E(\mathfrak{A}) \simeq O(E)$$

where $O(E)$ is the full orthogonal group of E; the orthogonal transformation determined by $\phi \in \mathrm{Aut}_E(\mathfrak{A})$ is its restriction to E. Our main result is:

PROPOSITION 1. *Let α be a periodic automorphism of $\mathfrak{A}$ with outer and inner invariants $p_0(\alpha)$, $\gamma(\alpha)$, $\epsilon(\alpha)$. Then α is conjugate to a Bogoliubov automorphism if and only if one of the following conditions holds:*

(i) $p_0(\alpha) = 1, 2, 3,...; \gamma(\alpha) = 1; \epsilon(\alpha) = \epsilon_1 * \cdots * \epsilon_n$,

(ii) $p_0(\alpha) = 2, 4, 6,...; \gamma(\alpha) = -1; \epsilon(\alpha) = \epsilon_1 * \cdots * \epsilon_n$.

Here, ϵ_j is the symmetric Bernoulli measure on a pair of conjugate qth roots of unity, $q = 1, 2, 3,...$, and $*$ denotes convolution.

It follows immediately that there are countably many conjugacy classes of periodic elements in $\mathrm{Aut}_E(\mathfrak{A})$.

In the statement of Proposition 1, let ϵ_j be supported by a pair of conjugate qth roots of unity, where $j = 1,..., n$; and let q be the least number with this property. The period of α (an invariant) is as follows: $qp_0(\alpha)$ in case (i); $2qp_0(\alpha)$ in case (ii).

It is clear that $\epsilon(\alpha)$, a probability measure in the set of qth roots of unity, is a *dyadic* measure, i.e., assigns to each qth root a dyadic rational.

When $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n$, the inner invariant is, modulo 2π, a binomial distribution wrapped around the unit circle, centered at the point $(1, 0)$.

Our proof of Proposition 1 depends on a classical result of Blattner concerning the inner automorphisms in $\mathrm{Aut}_E(\mathfrak{A})$, and on the more constructive proof of Blattner's theorem by de la Harpe and Plymen [5].

2. THE OUTER AND INNER INVARIANTS OF CONNES

Let $\mathrm{Aut}(\mathfrak{A})$ be the group of all automorphisms of $\mathfrak{A}$, let $\mathrm{Inn}(\mathfrak{A})$ be the normal subgroup of inner automorphisms, and let

$$\mathrm{Out}(\mathfrak{A}) = \mathrm{Aut}(\mathfrak{A})/\mathrm{Inn}(\mathfrak{A})$$

be the quotient group.

Let W be a unitary element in $\mathfrak{A}$. Then Ad W is the inner automorphism defined by

$$(\mathrm{Ad}\ W)T = WTW^{-1}$$

for all T in $\mathfrak{A}$.

Let $\alpha \in \mathrm{Aut}\ \mathfrak{A}$. The number $p_0(\alpha)$ is defined as follows:

$$\{n \in \mathbb{Z},\ \alpha^n \in \mathrm{Inn}\ \mathfrak{A}\} = p_0(\alpha)\mathbb{Z} \qquad \text{and} \qquad p_0(\alpha) \in \mathbb{N}.$$

For each α, $p_0(\alpha)$ is an integer called the *outer period* of α; it is zero if all the non-zero powers of α are outer.

We have $\alpha^{p_0} = \mathrm{Ad}\ U$. For all $\sigma \in \mathfrak{A}$ we have

$$\sigma(\mathrm{Ad}\ U)\,\sigma^{-1} = \mathrm{Ad}\ \sigma(U).$$

Hence

$$\mathrm{Ad}\ \alpha(U) = \alpha(\mathrm{Ad}\ U)\,\alpha^{-1} = \alpha\alpha^{p_0}\alpha^{-1} = \alpha^{p_0} = \mathrm{Ad}\ U.$$

Therefore $U/\alpha(U)$ lies in the center of $\mathfrak{A}$. Consequently

$$\alpha(U) = \gamma(\alpha)U,$$

where $\gamma(\alpha)$ is a complex number of modulus 1. The number $\gamma(\alpha)$ is clearly independent of the choice of U such that $\alpha^{p_0(\alpha)} = \mathrm{Ad}\ U$. Now

$$\begin{aligned} \gamma(\alpha)^{p_0(\alpha)}U &= \alpha^{p_0(\alpha)}(U) \\ &= UUU^{-1} \\ &= U, \end{aligned}$$

therefore

$$\gamma(\alpha)^{p_0(\alpha)} = 1.$$

DEFINITION. α and β in Aut $\mathfrak{A}$ are called *outer conjugate* if there exists σ in Aut $\mathfrak{A}$ such that β and $\sigma\alpha\sigma^{-1}$ have the same image in Out $\mathfrak{A} = \mathrm{Aut}\ \mathfrak{A}/\mathrm{Inn}\ \mathfrak{A}$.

Let $\tilde{\alpha}$ be the coset of α in Out $\mathfrak{A}$. Clearly, α and β are outer conjugate if and only if $\tilde{\alpha}$ and $\tilde{\beta}$ are conjugate in Out $\mathfrak{A}$.

DEFINITION. $(p_0(\alpha), \gamma(\alpha))$ is the *outer invariant* of α.

PROPOSITION 2 (Connes). *Let $\alpha, \beta \in \mathrm{Aut}\ \mathfrak{A}$. Then α and β are outer conjugate if and only if $p_0(\alpha) = p_0(\beta)$, $\gamma(\alpha) = \gamma(\beta)$.*

Proof. Connes [4].

Let

$$\alpha^{p_0(\alpha)} = \mathrm{Ad}\, U,$$
$$\alpha(U) = \gamma(\alpha)\, U,$$
$$p_m = p_0 \cdot (\text{order}\, \gamma),$$
$$\alpha^{p_m} = \mathrm{Ad}\, V,$$

so that we may take $V = U^{(\text{order}\,\gamma)}$. The natural number p_m is called the ***minimal period*** of α. Let

$$\beta = \alpha^{p_m}$$

and suppose further that β is *periodic*, i.e.,

$$\beta^k = 1$$

for some $k = 1, 2, 3,\ldots$. Let F be the spectral measure of V. Since V is unitary, F is a spectral measure defined on the Borel sets of the unit circle S^1. Since $\beta^k = I$, it follows that $V^k = \lambda_0 \in S^1$. We define the *inner invariant* of α to be

$$\epsilon(\alpha) = \tau \circ F,$$

where τ is the trace on the hyperfinite II_1 factor $\mathfrak{A}$. Then $\epsilon(\alpha)$ is a probability measure on the set $\{a_1, \ldots, a_k\}$ of kth roots of λ_0 and is determined up to a rotation of S^1. It is easy to see that two inner automorphisms α and β are conjugate if and only if $\epsilon(\alpha) = \epsilon(\beta)$ and that all probability measures on S^1 which have support contained in the kth roots of some $\lambda_0 \in S^1$ arise as $\epsilon(\alpha)$.

PROPOSITION 3 (Connes). *Two periodic automorphisms $\alpha, \beta \in \mathrm{Aut}\, \mathfrak{A}$ are conjugate if and only if they have the same outer and inner invariants, i.e.,*

$$p_0(\alpha) = p_0(\beta), \qquad \gamma(\alpha) = \gamma(\beta), \qquad \epsilon(\alpha) = \epsilon(\beta).$$

Proof. Connes [4].

Let $\epsilon = \epsilon(\alpha)$ and let $\hat{\epsilon}$ be the Fourier transform of ϵ. The dual group of the circle group S^1 is the integer group $\mathbb{Z}$, and $\hat{\epsilon}$ is by definition

$$\hat{\epsilon}(n) = \int z^n \, d\epsilon(z).$$

Now

$$V = \int z \, dF(z),$$
$$V^n = \int z^n \, dF(z),$$
$$\tau(V^n) = \int z^n \, d\epsilon(z)$$

so that, for all $n \in \mathbb{Z}$,

$$\hat{\epsilon}(n) = \tau(V^n).$$

It is technically easier to work with the Fourier transform $\hat{\epsilon}$ rather than ϵ; we shall compute $\hat{\epsilon}(\alpha)$ whenever α is a periodic Bogoliubov automorphism, and subsequently determine $\epsilon(\alpha)$.

3. Computation of the Outer Invariant

Following the notation of Blattner [2], let G^+ be the group of all T in $O(E)$ such that

(i) $I - T$ is Hilbert–Schmidt,

(ii) $\dim\{e \in E: Te = -e\}$ is even,

and let G^- be the set of all T in $O(E)$ such that

(iii) $I + T$ is Hilbert–Schmidt,

(iv) $\dim\{e \in E: Te = e\}$ is odd.

Let

$$\mathrm{Inn}_E(\mathfrak{A}) = \mathrm{Inn}(\mathfrak{A}) \cap \mathrm{Aut}_E(\mathfrak{A}).$$

If $R \in O(E)$, then $\Gamma(R)$ is the unique element in $\mathrm{Aut}(\mathfrak{A})$ such that

$$\Gamma(R)\, x_1 \cdots x_n = Rx_1 \cdots Rx_n$$

for all $x_1 ,..., x_n \in E$, $n = 1, 2, 3,...$. The map $R \mapsto \Gamma(R)$ is an isomorphism of $O(E)$ onto $\mathrm{Aut}_E(\mathfrak{A})$. Let N be the normal subgroup of $O(E)$ corresponding to $\mathrm{Inn}_E(\mathfrak{A})$, so that $R \in N$ if and only if $\Gamma(R) \in \mathrm{Inn}_E(\mathfrak{A})$. The following result of Blattner is crucial:

Proposition 4. $N = G^+ \cup G^-$.

We are interested in Bogoliubov automorphisms α with outer period p_0. To simplify notation we shall write $p = p_0$. In this case we have

$$\alpha = \Gamma(R),$$
$$\alpha^p = \mathrm{Ad}\, U,$$
$$R^p \in G^+ \cup G^-.$$

Our first task is, accordingly, to identify those R in $O(E)$ such that $R^p \in G^+ \cup G^-$. Let

$$\begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} = S(\lambda).$$

Then

$$
\begin{aligned}
\| 1 - S(\lambda)\|_2^2 &= \operatorname{Tr}\{1 - S(\lambda)\}^*\{1 - S(\lambda)\} \\
&= \operatorname{Tr}\begin{pmatrix} 1 - \cos\lambda & -\sin\lambda \\ \sin\lambda & 1 - \cos\lambda \end{pmatrix}\begin{pmatrix} 1 - \cos\lambda & \sin\lambda \\ -\sin\lambda & 1 - \cos\lambda \end{pmatrix} \\
&= \operatorname{Tr}\begin{pmatrix} 2 - 2\cos\lambda & 0 \\ 0 & 2 - 2\cos\lambda \end{pmatrix} \\
&= 4(1 - \cos\lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
\| 1 + S(\lambda)\|_2^2 &= \operatorname{Tr}\{1 + S(\lambda)\}^*\{1 + S(\lambda)\} \\
&= \operatorname{Tr}\begin{pmatrix} 1 + \cos\lambda & \sin\lambda \\ -\sin\lambda & 1 + \cos\lambda \end{pmatrix}\begin{pmatrix} 1 + \cos\lambda & -\sin\lambda \\ \sin\lambda & 1 + \cos\lambda \end{pmatrix} \\
&= \operatorname{Tr}\begin{pmatrix} 2 + 2\cos\lambda & 0 \\ 0 & 2 + 2\cos\lambda \end{pmatrix} \\
&= 4(1 + \cos\lambda).
\end{aligned}
$$

Let $R \in O(E)$. Suppose that $I - R^p$ or $I + R^p$ is compact. By the spectral theory of compact operators on real Hilbert space, there exist orthogonal subspaces E^+, E^-, E_1, E_2, E_3,..., such that

$$
\begin{aligned}
E &= E^+ \oplus E^- \bigoplus_n E_n, \\
R &= (1) \oplus (-1) \bigoplus_n S(\lambda_n), \\
R^p &= (1) \oplus (-1)^p \bigoplus_n S(p\lambda_n), \\
\dim E^+ &= 0 \quad \text{or} \quad 1, \\
\dim E^- &= 0 \quad \text{or} \quad 1, \\
\dim E_n &= 2.
\end{aligned}
$$

Suppose first that $R^p \in G^+$. Then $\sum (1 - \cos p\lambda_n) < \infty$. Subject to this condition, the following four possibilities arise:

(1) $R^p = \oplus_n S(p\lambda_n)$,

(2) $R^p = (1) \oplus_n S(p\lambda_n)$,

(3) $R^p = (-1)^p \oplus_n S(p\lambda_n)$, p even,

(4) $R^p = (1) \oplus (-1)^p \oplus_n S(p\lambda_n)$, p even.

When $R^p \in G^-$, we have $\sum (1 + \cos p\lambda_n) < \infty$. Subject to this condition, the following three possibilities arise:

(5) $R^p = (1) \oplus_n S(p\lambda_n)$,

(6) $R^p = (-1)^p \oplus_n S(p\lambda_n)$, p even,

(7) $R^p = (1) \oplus (-1)^p \oplus_n S(p\lambda_n)$, p odd.

In each case (1)–(7), we are going to compute a unitary operator U such that

$$\alpha^p = \mathrm{Ad}\, U$$

and then compute $\gamma(\alpha)$ directly from the defining equation

$$\alpha(U) = \gamma(\alpha)U.$$

We consider cases (1)–(4), corresponding to $R^p \in G^+$, simultaneously. Here, we have

(1) $R^p = \oplus_n S(p\lambda_n)$,

(2) $R^p = (1) \oplus_n S(p\lambda_n)$,

(3) $R^p = (1) \oplus_n S(p\lambda_n)$, p even,

(4) $R^p = (1) \oplus (1) \oplus_n S(p\lambda_n)$, p even

subject to

$$\sum (1 - \cos p\lambda_n) < \infty. \tag{3.1}$$

Let r_n be the unique integer such that

$$-\pi < p\lambda_n + 2r_n\pi \leqslant \pi$$

and set

$$\psi_n = p\lambda_n + 2r_n\pi.$$

Condition (3.1) is now $\sum (1 - \cos \psi_n) < \infty$ and since $-\pi < \psi_n \leqslant \pi$ this means that $\psi_n \to 0$. Hence there exists n_0 such that $1 - \cos \psi_n/2 \leqslant 1 - \cos \psi_n$ whenever $n \geqslant n_0$. Hence

$$\sum (1 - \cos \psi_n/2) < \infty.$$

Set

$$v_n = \cos \psi_n/2 - \sin \psi_n/2 \cdot e_{2n-1}e_{2n},$$

where $\{e_{2n-1}, e_{2n}\}$ is an orthonormal basis in the Euclidean plane E_n. Set

$$U = \prod v_n = v_1v_2v_3 \cdots.$$

Then the infinite product $\prod v_n$ is strongly convergent in the unitary group of $\mathfrak{A}$, and

$$\alpha^p = \Gamma(R^p) = \text{Ad } U$$

by Lemma 6(ii) in [5].

It is easy to see that $e_{2n-1}e_{2n}$ is an eigenvector of $\Gamma(R)$ with eigenvalue 1:

$$\begin{aligned}\Gamma(R)\, e_{2n-1}e_{2n} &= Re_{2n-1} \cdot Re_{2n} \\ &= (\cos \lambda_n \cdot e_{2n-1} + \sin \lambda_n \cdot e_{2n})(-\sin \lambda_n \cdot e_{2n-1} + \cos \lambda_n \cdot e_{2n}) \\ &= \cos^2 \lambda_n \cdot e_{2n-1}e_{2n} - \sin^2 \lambda_n e_{2n}e_{2n-1} \\ &= e_{2n-1}e_{2n}\,.\end{aligned}$$

Hence $\Gamma(R)\, v_n = v_n$. By strong continuity of $\Gamma(R)$ we have

$$\Gamma(R) \prod v_n = \prod v_n\,.$$

This equation says that $\alpha(U) = U$. Hence $\gamma(\alpha) = 1$ in cases (1)–(4).

It remains to consider cases (5)–(7). Here, we have

(5) $R^p = (1) \oplus_n S(p\lambda_n)$,

(6) $R^p = (1) \oplus_n S(p\lambda_n)$, p even,

(7) $R^p = (1) \oplus (-1) \oplus_n S(p\lambda_n)$, p odd,

subject to

$$\sum (1 + \cos p\lambda_n) < \infty. \tag{3.2}$$

Let s_n be the unique integer such that

$$-\pi < p\lambda_n + 2s_n\pi - \pi \leqslant \pi$$

and set

$$\phi_n = p\lambda_n + 2s_n\pi - \pi.$$

Then $1 + \cos\ p\lambda_n = 1 + \cos(\phi_n + \pi) = 1 - \cos\ \phi_n$ so condition (3.2) is

$$\sum (1 - \cos \phi_n) < \infty.$$

Since $-\pi < \phi_n \leqslant \pi$, this means that $\phi_n \to 0$. Hence

$$\sum (1 - \cos \phi_n/2) < \infty.$$

Set

$$u_n = \cos \phi_n/2 - \sin \phi_n/2 \cdot e_{2n-1}e_{2n}\,,$$

$$U = \prod u_n = u_1u_2u_3 \cdots.$$

Then $\prod u_n$ is strongly convergent in the unitary group of $\mathfrak{A}$, by Lemma 6(ii) in [5].

Let e^+ (resp. e^-) be a unit vector in E^+ (resp. E^-). We claim that, in cases (5)–(7), a unitary U such that $\alpha^p = \mathrm{Ad}\, U$ is given by

(5) $U = e^+ \cdot \prod u_n$,

(6) $U = e^- \cdot \prod u_n$,

(7) $U = e^+ \cdot \prod u_n$.

This claim depends on the following standard observations:

(i) $\mathrm{Ad}\, e^+ = -\rho$, where ρ denotes mirror reflection in the orthogonal complement of E^+;

(ii) $\mathrm{Ad}\, e^- = -\sigma$, where σ denotes mirror reflection in the orthogonal complement of E^-;

(iii) $\mathrm{Ad}\, u_n$ determines a rotation through $p\lambda_n - \pi$ in the Euclidean plane E_n;

(iv) e^+ and e^- commute with $\prod u_n$.

In cases (5) and (7), we compute $\gamma(\alpha)$ as follows:

$$\begin{aligned}\alpha(U) &= \Gamma(R)\, U\\ &= Re^+ \cdot \Gamma(R) \prod u_n\\ &= e^+ \cdot \prod u_n\\ &= U\end{aligned}$$

so that $\gamma(\alpha) = 1$. In case (6), we have

$$\begin{aligned}\alpha(U) &= \Gamma(R)\, U\\ &= Re^- \cdot \Gamma(R) \prod u_n\\ &= -e^- \cdot \prod u_n\\ &= -U\end{aligned}$$

so that $\gamma(a) = -1$.

Collating the information thus obtained and quoting Proposition 2 of Connes, we have

LEMMA 1. *Let α be a periodic automorphism of $\mathfrak{A}$ with outer invariant $p_0(\alpha)$, $\gamma(\alpha)$. Then α is outer conjugate to a Bogoliubov automorphism if and only if one of the following conditions holds:*

(i) $p_0(\alpha) = 1, 2, 3, \ldots$; $\gamma(\alpha) = 1$,

(ii) $p_0(\alpha) = 2, 4, 6, \ldots$; $\gamma(\alpha) = -1$.

4. Computation of the Inner Invariant

We proceed to refine Lemma 1 by computing the inner invariant $\epsilon(\alpha)$ of a periodic Bogoliubov automorphism.

The notations are exactly the same as in previous sections.

We consider first cases (1)–(4). Here, we have

$$\alpha^p = \operatorname{Ad} U,$$
$$U = \prod v_n,$$
$$\gamma(\alpha) = 1,$$
$$p_m = p,$$
$$\beta = a^{p_m} = \operatorname{Ad} U,$$

where $p_m = p(\text{order } \gamma(\alpha)) =$ minimal period of α. We suppose further that β is periodic, i.e.,

$$\beta^k = 1$$

for some $k = 1, 2, 3, \ldots$. We have

$$\operatorname{Ad} U^k = 1$$

and so $U^k = \lambda_0 \in S^1$. We take, without loss of generality, $\lambda_0 = 1$. Now the v_n mutually commute, the product $\prod v_n$ is strongly convergent, and so we have

$$\prod v_n^k = 1.$$

Taking the trace, we get

$$\prod \tau(v_n^k) = 1.$$

Therefore $\tau(v_n^k) = 1$, and so $v_n^k = 1$, $n = 1, 2, 3, \ldots$. Hence $k\psi_n/2 = 2l_n\pi$ with $l_n \in \mathbb{Z}$. Thus

$$\psi_n = 4l_n\pi/k.$$

Since $\psi_n \to 0$ as $n \to \infty$, there exists a finite N such that $\psi_n = 0$ whenever $n > N$. Thus U is a finite product:

$$U = v_1 \cdots v_N .$$

Let $Q_n^+ = (1 + ie_{2n-1})/2$, $Q_n^- = (1 - ie_{2n-1}e_{2n})/2$. Then $Q_n^+ + Q_n^- = 1$ and $(Q_n^+)^* = Q_n^+ = (Q_n^+)^2$ so that Q_n^+ and Q_n^- are orthogonal projections in $\mathfrak{A}$. The spectral resolution of v_n is clearly

$$v_n = \exp(i\psi_n/2) \cdot Q_n^+ + \exp(-i\psi_n/2) \cdot Q_n^-.$$

Therefore

$$U = \sum \exp \frac{i}{2} (\pm\psi_1 \cdots \pm\psi_N) Q_1^{\pm} \cdots Q_N^{\pm}.$$

Hence

$$\begin{aligned}\hat{\epsilon}(r) &= \tau(U^r)\\ &= 2^{-N} \sum \exp \frac{ir}{2} (\pm\psi_1 \pm \cdots \pm \psi_N)\\ &= \prod_{n=1}^{N} \cos(r\psi_n/2). \end{aligned} \tag{4.1}$$

We now consider cases (5) and (7). Here,

$$\begin{aligned}\alpha^p &= \mathrm{Ad}\, U,\\ U &= e^+ \cdot \prod u_n\,,\\ \gamma(\alpha) &= 1,\\ p_m &= p,\\ \beta &= \alpha^{p_m} = \mathrm{Ad}\, U,\\ \beta^k &= 1.\end{aligned}$$

We must have

$$(e^+)^k = 1, \qquad u_n{}^k = 1$$

so that k is even and

$$k\phi_n/2 = 2l_n\pi$$

for some $l_n \in \mathbb{Z}$. But $\phi_n \to 0$ as $n \to \infty$; therefore U is a finite product

$$U = e^+ \cdot u_1 \cdots u_N\,.$$

Let $Q^+ = (1 + e^+)/2$ and $Q^- = (1 - e^+)/2$. Then U has the spectral resolution

$$U = (Q^+ - Q^-) \prod_{n=1}^{N} \{\exp(i\phi_n/2) \cdot Q_n{}^+ + \exp(-i\phi_n/2) \cdot Q_n{}^-\}.$$

Hence

$$U^r = (Q^+ + (-1)^r Q^-) \prod \{\exp(ir\phi_n/2) \cdot Q_n{}^+ + \exp(-ir\phi_n/2) \cdot Q_n{}^-\}$$

and

$$\begin{aligned}\hat{\epsilon}(r) &= \tau(U^r)\\ &= \tfrac{1}{2}\{1 + (-1)^r\}\, 2^{-N} \sum \exp \frac{ir}{2}(\pm\phi_1 \pm \cdots \pm \phi_N)\\ &= \cos^2 r\pi/2 \cdot \prod_{n=1}^{N} \cos(r\phi_n/2) \qquad (4.2)\end{aligned}$$

since $\cos^2 r\pi/2 = \frac{1}{2}(1 + \cos r\pi)$.

Finally, consider case (6). Here,

$$\begin{aligned}\alpha^p &= \mathrm{Ad}\, U,\\ U &= e^- \cdot \prod u_n\,,\\ \gamma(\alpha) &= -1,\\ p_m &= 2p,\\ \beta &= \alpha^{p_m} = \mathrm{Ad}\, U^2,\\ \beta^k &= 1.\end{aligned}$$

In this case $p_m = 2p$ and so we set $V = U^2$ and compute $\tau(V^r)$. Now

$$V = \prod u_n{}^2$$

and again V is a finite product:

$$\begin{aligned}V &= u_1{}^2 \cdots u_N{}^2,\\ V^r &= \sum \exp ir(\pm\phi_1 \pm \cdots \pm \phi_N),\\ \hat{\epsilon}(r) &= \tau(V^r)\\ &= \prod_{n=1}^{N} \cos r\phi_n\,. \qquad (4.3)\end{aligned}$$

In formula (4.1) we have $k\psi_n/2 \equiv 0 \bmod 2\pi$; in (4.2) we have k even and $k\phi_n/2 \equiv 0$; in (4.3) we have $k\phi_n \equiv 0$. Note that (4.1) and (4.2) hold when $\gamma(\alpha) = +1$; (4.3) holds when $\gamma(\alpha) = -1$. We have established:

LEMMA 2. *Let α be a periodic Bogoliubov automorphism of $\mathfrak{A}$. Then the Fourier transform of the inner invariant $\epsilon(\alpha)$ is a finite product of cosine terms:*

$$\hat{\epsilon}(r) = \prod \cos r\theta_n \qquad r \in \mathbb{Z},$$

where $k\theta_n \equiv 0 \bmod 2\pi$ for some natural number k.

We proceed to deduce Proposition 1 from this result. Now $\exp i\theta_n$ is a kth root of unity, and all such roots arise from suitable choices of p_0 and $\{\lambda_n\}$. Let μ be the symmetric Bernoulli measure with support $\{\exp i\theta_n, \exp(-i\theta_n)\}$. Then

$$\hat{\mu}(r) = \int z^r \, d\mu(z) = \tfrac{1}{2} \exp ir\theta_n + \tfrac{1}{2} \exp(-ir\theta_n) = \cos r\theta_n .$$

Taking the inverse Fourier transform, we find that $\epsilon(\alpha)$ is a finite convolution product

$$\epsilon_1 * \cdots * \epsilon_n ,$$

where ϵ_j is the symmetric Bernoulli measure on a pair of conjugate qth roots of unity. Proposition 1 now follows from Lemmas 1 and 2, and Proposition 3 of Connes.

5. EXAMPLES

Let ϵ_0 be the symmetric Bernoulli measure on $\{-1, +1\}$. If η is the symmetric Bernoulli measure on $\{-i, +i\}$, then

$$\eta * \eta = \epsilon_0 .$$

The probability measure with support $\{1\}$ will be denoted 1.

EXAMPLE 1.

$$\begin{aligned}
&R = (1) \oplus S(\pi/2) \oplus S(\pi/2) \oplus S(\pi/2) \oplus \cdots ,\\
&R^2 = (1) \oplus (-I),\\
&\alpha^2 = \Gamma(R^2) = \operatorname{Ad} e^+,\\
&\alpha(e^+) = Re^+ = e^+,\\
&\gamma(\alpha) = 1,\\
&p_m = p_0 = 2,\\
&\epsilon(\alpha) = \epsilon_0 ,\\
&\alpha^4 = 1.
\end{aligned}$$

Invariants. $(p_0, \gamma, \epsilon) = (2, 1, \epsilon_0)$.

EXAMPLE 2.

$$\begin{aligned}
&R = (-1) \oplus S(\pi/2) \oplus S(\pi/2) \oplus S(\pi/2) \oplus \cdots ,\\
&\alpha^2 = \Gamma(R^2) = \operatorname{Ad} e^-,\\
&\alpha(e^-) = Re^- = -e^-,\\
&\gamma(\alpha) = -1,\\
&p_m = 2p_0 = 4,\\
&\alpha^4 = 1.
\end{aligned}$$

Invariants. $(2, -1, 1)$.

EXAMPLE 3.

$$\begin{aligned}
&\alpha = \mathrm{Ad}\, e, \qquad \|e\| = 1, e \in E\\
&R = (1) \oplus (-I),\\
&\alpha(e) = Re = e,\\
&\gamma(\alpha) = 1,\\
&\epsilon = \epsilon_0,\\
&\alpha^2 = 1.
\end{aligned}$$

Invariants. $(1, 1, \epsilon_0)$.

EXAMPLE 4.

$$\begin{aligned}
&\alpha' = \mathrm{Ad}\, e_1e_2e_3e_4, \qquad \{e_1, e_2, e_3, e_4\} \text{ orthonormal in } E,\\
&R' = (-1) \oplus (-1) \oplus (-1) \oplus (-1) \oplus (I),\\
&\alpha'(e_1e_2e_3e_4) = R'e_1 \cdot R'e_2 \cdot R'e_3 \cdot R'e_4 = e_1e_2e_3e_4,\\
&\gamma(\alpha') = 1,\\
&\epsilon = \epsilon_0,\\
&\alpha'^2 = 1.
\end{aligned}$$

Invariants. $(1, 1, \epsilon_0)$.

In examples 3 and 4 we see that α and α' are conjugate involutions, even though R is not conjugate in $O(E)$ to R' because R and R' have different spectral multiplicities. Indeed, $R \in G^-$ and $R' \in G^+$.

Since $\epsilon_0 * \epsilon_0 = \epsilon_0$, it follows that

$$\eta * \eta = \eta * \eta * \eta * \eta = \eta * \eta * \eta * \eta * \eta * \eta = \cdots,$$

where η is the symmetric Bernoulli measure on $\{+i, -i\}$, a conjugate pair of fourth roots of unity. There is therefore no "unique factorization" among the convolution products $\epsilon_1 * \cdots * \epsilon_n$.

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Note added in proof. The factorization problem among the convolution products turns out to be a delicate matter, and is studied in some detail in [7].

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